Matrix-Vector Products

Definition: For a $m \times n$ matrix A and a vector $\mathbf{x}$ in $\mathbb{R}^n$ we define the matrix-vector product

$$A\mathbf{x} = \begin{array}{c} \begin{array}{cccc} \vec{v}_1 & \vec{v}_2 & \quad & \vec{v}_n \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} \tag{1}$$

or equivalently

$$\boxed{A\mathbf{x} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n} \tag{2}$$

where the vectors

$$\mathbf{v}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \quad \cdots \quad \mathbf{v}_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} \tag{3}$$

are called the column vectors of A.

Note 1: Equation (2) says that $A\mathbf{x}$ is a *linear combination* of the column vectors of A where the coefficients are determined by the components of the vector $\mathbf{x}$.

Example 1: Let $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 2 & -3 \\ 4 & -1 & 0 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. Calculate $A\mathbf{x}$.

$$A\vec{x} = 1\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + (-1)\begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} + (1)\begin{bmatrix} 0 \\ -3 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1+0 \\ 2-2-3 \\ 4+1+0 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$$

Example 2: Calculate $I_3 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. What do you observe? $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$I_3 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = (1)\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Note: $I_3 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

Proposition 1: For any vector $\mathbf{x}$ in $\mathbb{R}^n$ we have $I_n\mathbf{x} = \mathbf{x}$.

Example 3: Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$ be the column vectors of I_4.

$$I_4 = \begin{array}{c} \begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Then $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix} \mathbf{e}_2 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = 0\begin{bmatrix} 1 \\ 5 \\ 9 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 6 \\ 10 \end{bmatrix} + 0\begin{bmatrix} 3 \\ 7 \\ 11 \end{bmatrix} + 0\begin{bmatrix} 4 \\ 8 \\ 12 \end{bmatrix}$

$$= \begin{bmatrix} 2 \\ 6 \\ 10 \end{bmatrix}$$

Proposition 2 (Poole 3.1b): Suppose the $m \times n$ matrix A has column vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ in $\mathbb{R}^m$. Then for any i in $1, \ldots, n$ we have

$$\boxed{A\mathbf{e}_i = \mathbf{v}_i} \quad \text{"column selection"} \tag{4}$$

where $\boxed{\mathbf{e}_i \text{ is the } i^{th} \text{ column vector of } I_n.}$

Definition: By definition of matrix-vector product, the $m \times n$ linear system in *vector form*

$$A\vec{x} = \boxed{x_1 \overset{\vec{v}_1}{\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}} + x_2 \overset{\vec{v}_2}{\begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}} + \cdots + x_n \overset{\vec{v}_n}{\begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}} \tag{5}$$

can be written in matrix-vector form as

$$\boxed{A\mathbf{x} = \mathbf{b}} \tag{6}$$

where

$$A = \begin{array}{c} \begin{array}{cccc} \vec{v}_1 & \vec{v}_2 & & \vec{v}_n \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \end{array}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \tag{7}$$

Example 4: Write the linear system given in matrix-vector form

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & -1 & -2 \\ 2 & 1 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} \tag{8}$$

in *vector form* and *equation form.*

Vector form

$$x_1 \underbrace{\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}}_{\vec{v}_1} + x_2 \underbrace{\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}}_{\vec{v}_2} + x_3 \underbrace{\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}}_{\vec{v}_3} = \underbrace{\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}}_{\vec{b}}$$

$$x_1 \vec{v}_1 + x_2 \vec{v}_2 + x_3 \vec{v}_3 = \vec{b}$$

Equation Form

$$x_1 + 2x_2 + 3x_3 = 1$$
$$x_1 - x_2 - 2x_3 = -1$$
$$2x_1 + x_2 + x_3 = -2$$